In-Plane Stresses in Edge Stiffened Swept Panels

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Swept panels under extension present a significant structural problem in modern aircraft design. In this paper, a simple and elegant analysis for swept panels is introduced with the case of a parallelogrammic panel bounded by flexible constant stress edge members. The stress function is assumed in two alternative infinite series in polar coordinates, one function with origin at the acute corner isolates the finite stress concentration at that corner, and the other centered at the obtuse corner separates out the singularity at that abtuse corner. The arbitrary constants in the series are determined by a collocation procedure for approximately satisfying the skew conditions along the relevant diagonal of the panel. The solutions converge rapidly in the region around their origin, but their convergence far from the corner is normally poor. Accurate results for the entire panel are obtained by combining the solutions with origins at the two corners, each solution contributing the data for the region where it is more rapidly convergent.

Nomenclature[‡§]

A_m, B_m	= arbitrary constants in the mth term in the stress function
$2a, 2b$	= lengths of panel edges
EI_f, EI_r	= flexural stiffness of flanges and ribs, respectively
F	= flange area
M	= stage of truncation of series
P, Q	= flange and rib load, respectively
q	= external shear applied to edge members of panel
t	= thickness of panel
x, y	= Cartesian coordinates
r, θ	= polar coordinates
σ_{AB}, σ_{BC} σ_{CD}, σ_{DA}	= constant axial stresses in edge members AB, BC CD, and DA, respectively
σ_{of}, σ_{or}	= constant axial stresses in flanges and ribs (introduced for skew-symmetric and skew-antisymmetric cases)
$\sigma_x, \sigma_y, \sigma_{xy}$	= direct and shear stresses in Cartesian coordinates
$\sigma_r, \sigma_\theta, \sigma_{r\theta}$	= direct and shear stresses in polar coordinates
ϕ	= Airy stress function

Subscripts

$F\&P$	= single letter indicates location (e.g. P_A), pairs of letters indicate edge along which the quantity varies (e.g. P_{AB})

I. Introduction

SWEPT panels are frequently used as structural elements in modern aircraft. The analysis of such swept panels, identifying the stress concentrations or singularities at the panel corners, and obtaining highly accurate information in the regions of large stresses, is very important for their efficient design. The corner function[1-3] (biharmonic functions in polar coordinates) can isolate singularities of the type $r^{-\lambda}$ $(o<\lambda<1)$ arising due to homogeneous boundary conditions on the adjacent edges of such panels. A basic stress system[2-4] of the same form can also be used to isolate stress concentrations or logarithmic singularities due to nonhomogeneous boundary conditions on these edges. The method has, in the past, been applied to the relatively simple rectangular and sector shapes.[5,6] It is the purpose, now, to develop their application to the more practical case of swept parallelogrammic (skew) panels. In the literature on swept field, Hemp's[7] work on the application of oblique coordinates and Quinlan's[8] development of λ – method are two significant analytical contributions. However their work does not explicity or accurately determine the corner stress concentration.

In this paper, an elegant analysis for swept panels is introduced with the simple case of extension of parallelogrammic panels bounded by flexible constant stress edge members. A finite stress concentration occurs at the acute corner, and a stress singularity occurs at the obtuse corner of such a panel. The stress function is assumed in two alternative infinite series in polar coordinates. The first series with origin of coordinates at the acute corner isolates the finite stress concentration at that corner, and the other centered at the obtuse corner separates out the singularity at the obtuse corner. For practical analysis the series are terminated after a few terms, the number of terms retained depending on the accuracy desired. The arbitrary constants in the series are determined by a simple collocation procedure to approximately satisfy the skew-symmetry or skew-antisymmetry conditions along the relevant diagonal of the panel.

Proposal for Combining Multiple Solutions to Cover the Field

A solution located at a corner is found to converge rapidly in the region around the origin, so that only a few terms in the series are sufficient to achieve a high degree of accuracy, but its convergence far from the corner can be very poor. It is proposed and demonstrated in this paper that if the two solutions located at obtuse and acute corners of the panel, each accurate in a part of the panel, are suitable matched and put together, accurate data for the entire panel can be derived with limited effort. This effective technique of combining multiple solutions, each with separate regions of accuracy, has possibilities for application in many difficult problems.[6] The method of analysis proposed in this paper finds application in the development of hybrid continuum-finite element formulations[9] for problems with more than one source of stress concentration or singularity. Numerical results are presented for a parallelogrammic panel with 30° sweep under skew-symmetric and skew-antisymmetric loadings. The quantities of design significance such as edge shear distribution and flange area variations are shown.

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‡Edge members in the longitudinal direction (along AB, CD) are termed flanges and those in the transverse direction (along BC, DA) are termed ribs, (Fig. 1).

§The solutions are characterized by a symbolic description, e.g. in the description $PAC(BD)$-04, P stands for solution in polar coordinates, A is the origin of coordinates, C refers to application of collocation for satisfying the boundary conditions, (BD) being the relevant boundary, and –04 indicates that the series are terminated at $m=M=4$ (up to and inclusive of $2m=4$).

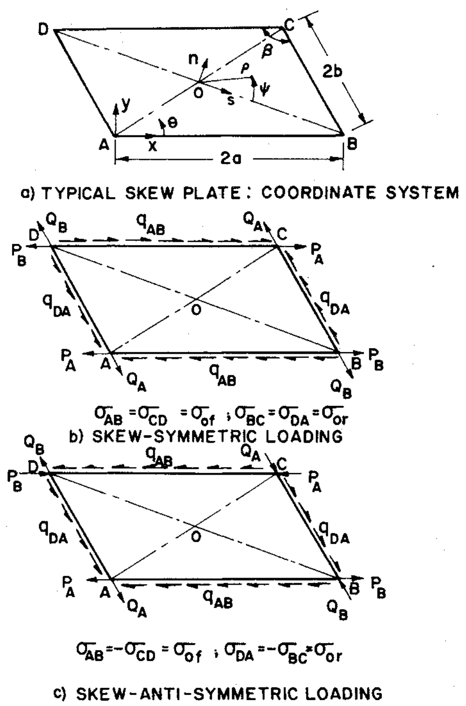

Fig. 1 Parallelogrammic panel.

II. Problem

The parallelogrammic panel shown in Fig. 1 is bounded by flexible members ($EI_f = EI_r = 0$) along all four edges AB, BC, CD and DA. The edge members are so tapered that under loads applied axially at their ends and along their lengths, they develop constant axial stresses σ_{AB}, σ_{BC}, σ_{CD}, σ_{DA}. Under a skew-symmetric applied load system, as in Fig. 1b, the internal stress field is skew-symmetric about any straight line passing through the center 0. Similarly, under the skew-antisymmetric applied load system shown in Fig. 1c, the stress field is skew-antisymmetric about any line through 0. It is convenient to consider the diagonals as "skew axes", and analyse only one half of the panel, either $ABOD$ or $BAOC$. The method of analysis for a half of the panel either $ABOD$ or $BAOC$ should be identical, but for the fact that the corner angle β is obtuse in the former and is acute in the later. Solutions are developed for the cases of skew-symmetric and skew-antisymmetric loadings, viz., $\sigma_{AB} = \pm\sigma_{CD} = \sigma_{of}$ and $\sigma_{AD} = \pm\sigma_{BC} = \sigma_{or}$. The state of stress due to general loading (Fig. 1a) can be obtained by combining suitable proportions of skew-symmetric and skew-antisymmetric stress field.

III. Analysis

With the assumptions of linear homogeneous isotropic elasticity, the stress field in $ABOD$ can be represented by an Airy stress function [10] governed by the biharmonic equation

$$\nabla^4 \phi = 0 \tag{1}$$

The boundary conditions on the external edges AB and AD meeting at the corner A are expressible as

$$\sigma_r = \sigma_{of} \text{ and } \sigma_\theta = 0 \text{ (along } AB) \ (\theta = 0) \tag{2a}$$

and

$$\sigma_r = \sigma_{or} \text{ and } \sigma_\theta = 0 \text{ (along } AD) \ (\theta = \beta) \tag{2b}$$

A stress function satisfying the governing differential equation, Eq. (1) and boundary conditions, Eqs. (2a and b) can be written down in an infinite series of polar-trigonometric biharmonic functions, and the series truncated at a suitable stage for numerical evaluation. That is,

$$\phi = \frac{\sigma_{of} r^2}{4}\left[\frac{(\beta-\theta)\sin 2\beta - \beta\sin 2(\beta-\theta)}{\beta \sin 2\beta}\right] + \frac{\sigma_{or} r^2}{4}\left[\frac{\theta \sin 2\beta - \beta \sin 2\theta}{\beta \sin 2\beta}\right]$$
$$+ \sigma_{of} a^2 \sum_{m=1,2,\ldots M} [A_m + B_m (r/a)^2](r/a)^{m\pi/\beta} \sin(m\pi\theta/\beta) \tag{3}$$

It is clearly seen that at the corner a finite stress arises from each of the two terms outside the infinite series, while a stress singularity of order $r^{\pi/\beta - 2}$ is identified by the first term in the series, viz., $A_1 (r/a)^{\pi/\beta} \sin(\pi\theta/\beta)$. The finite contribution occurs irrespective of the angle A, but the stress singularity occurs for obtuse angle A $(\pi/2 < \beta < \pi)$, and disappears when A is acute $(0 < \beta < \pi/2)$.

The arbitrary constants A_m and B_m are to be determined by satisfying the skew conditions along the diagonal BD. These conditions are [11]

$$\phi(s) = \pm\phi(-s)$$
$$\phi_n(s) = \mp\phi_n(-s)$$
$$\nabla^2\phi(s) = \pm\nabla^2\phi(-s)$$
$$\nabla^2\phi_n(s) = \mp\nabla^2\phi_n(-s) \tag{4}$$

where the alternative signs on the right hand side refer to skew-symmetric and skew-anti-symmetric fields, respectively.

The skew conditions in Eq. (4), can be satisfied by a suitable technique, preference being given to a method which is simple and capable of yielding accurate results with limited effort. It is easy to see that a collocation technique with equidistant or equiangular stations is simple for the type of skew conditions in Eq. (4). It has been found that for the present problem such collocation is also satisfactory from the point of view of accuracy. Equidistant collocation is carried out in the examples worked out. Thus, for collocation of skew conditions at the center of the diagonal, and at $(k=1)$ pairs of points on both sides of the center 0 on the diagonal, the infinite series in Eq. (3) is terminated at $m = M = 2k$. The $4k$ simultaneous equations in A_m, B_m are generated by satisfying Eq. (4) at k number of points on the diagonal. The parameters A_m, B_m are evaluated by the solution of these simultaneous equations.

In a practical design problem, one is interested in flange area distribution for flange design, and in the edge shear distribution to ensure a proper sheet to flange connection. The results of design significance can be obtained after the arbitrary constants in the infinite series are determined for any stage of truncation (M) of the series. The edge shear distribution along flange AB $(\theta = 0)$ is

$$\frac{\sigma_{xy}}{\sigma_{of}} = \left[\frac{\sin 2\beta - 2\beta\cos 2\beta}{4\beta \sin 2\beta}\right] + \frac{\sigma_{or}}{\sigma_{of}}\left[\frac{2\beta - \sin 2\beta}{4\beta\sin 2\beta}\right] + \sum_{m=1,2,\ldots,M}\left[\left(\frac{m\pi}{\beta} - 1\right)A_m + \left(\frac{m\pi}{\beta} + 1\right)B_m\left(\frac{x}{a}\right)^2\right]$$
$$\times\ (m\pi/\beta)\ (x/a)^{(m\pi/\beta - 2)} \tag{5}$$

The flange area distribution can be obtained by using the diffusion equation along flange AB. That is

$$\sigma_{of} F_{AB} = P_{AB} = P_A + \int_0^x (q_{AB} - q)\,dx \tag{6}$$

Table 1 Parallelogrammic panels with flexible constant stress flanges and inextensional end ribs: shear stress along flange & flange area $\beta = 120°$, $a/b = 2$

Solution x/a	0	0.1	0.5	1.0	1.5	1.9	2.0
	Shear Stress σ_{xy}/σ_{of} along flange AB						
(A) Skew-symmetric loading: $P_A = P_B = P_C = P_D$; all $Q=0$, all $q=0$ (see Fig. 1a)							
PAC (BD)-04	Infinity	0.85406	0.17131	−0.12432	−0.30884	0.16818	0.58860
PAC (BD)-08	Infinity	0.85784	0.15657	−0.11933	−0.38135	0.29830	1.6040
PAC(BD)-16	Infinity	0.85813	0.15650	−0.11922	−0.37623	0.070406	5.8815
PBC (AC)-04	−6.7298	−2.9397	0.11682	−0.12056	−0.35791	−0.49851	−0.52741
PBC (AC)-08	−49.152	−8.2818	0.17859	−0.12016	−0.36234	−0.50072	−0.52741
PBC (AC)-16	−250.94	−1.4012	0.17703	−0.11982	−0.36500	−0.50201	−0.52741
(B) Skew-antisymmetric loading: $P_A = P_B = -P_C = -P_D$; all $Q=0$, all $q=0$ (see Fig. 1b)							
PAC (BD)-04	Infinity	0.30961	0.056086	0.021232	−0.011746	−0.17184	−0.27523
PAC(BD)-08	Infinity	0.30856	0.058413	0.029398	−0.10132	−0.60801	−0.0620
PAC (BD)-16	Infinity	0.30831	0.058620	0.029568	−0.096275	−1.8761	−14.551
PBC (AC)-04	12.408	−5.8667	0.19306	0.044662	−0.10747	−0.43202	−0.52741
PBC (AC)-08	−18.641	−1.5624	0.076338	0.043312	−0.10666	−0.43166	−0.52741
PBC(AC)-16	−70.953	−0.083713	0.077317	0.038205	−0.10254	−0.42961	−0.52741

Table 2 Swept panels with flexible constant stress flanges: convergence of strength of singularity at obtuse corner[a]

a) Skew-symmetric loading: $P_A = P_B = P_C = P_D$; all $Q=0$, all $q=0$ (See Fig. 1) $a/b = 1$ (Rhombus)				
	Coefficients A_I			
Solutions	$\beta = 95°$	120°	150°	170°
PASC(BD)b	1.6227(1)	0.48712(2)	0.43885(3)	0.47040(4)
PASC(BD)	1.6363(2)	0.48757(4)	0.43880(6)	0.46812(6)
PASC(BD)	1.6362(3)	0.48757(6)	0.43874(9)	0.42917(11)
b) Skew-symmetric loading: $P_A = P_B = P_C = P_D$; all $Q=0$; all $q=0$ $\beta = 120°$				
	Coefficient A_I			
Solutions	$a/b = 4.0$	2.0	0.5	0.25
PAC(BD)-04	0.30323	0.44284	0.42155	0.45075
PAC(BD)-08	0.31833	0.44564	0.41763	0.41336
PAC(BD)-16	0.31262	0.44580	0.41740	0.41084
c) Skew-anti-symmetric loading: $P_A = P_B = -P_C = -P_D$; all $Q=0$; all $q=0$ $\beta = 120°$				
	Coefficient A_I			
Solution $a/b=$	4.0	2.0	0.5	0.25
PAC(BD)-04	0.15922	0.20826	0.40806	0.46162
PAC(BD)-08	0.14637	0.20651	0.40403	0.41383
PAC(BD)-16	0.14572	0.20636	0.40380	0.41303

[a] The strengths of singular stresses as corner A, are proportional to the coefficient A_I in the stress function Eq. (3).
[b] The order of solution M is indicated in the parenthesis following the value of the coefficient A_I.

Thus, the flange area variation is given by

$$F_{AB}(x) = \left\{P_A + \int_0^x q_{AB}dx + t\left[\{\phi_y\}_{\substack{x=x\\y=0}} - \{\phi_y\}_{\substack{x=0\\y=0}}\right]\right/\sigma_{of} \quad (7)$$

IV. Numerical Examples

The method of analysis is illustrated by numerical examples for the skew-symmetric and skew-antisymmetric loading of the panel. In all the cases considered in numerical examples the panels are bounded by flexible constant stress flanges along AB, CD, and flexible inextensional members along BC, DA. Convergence of solution (up to $M=16$) for a few panel sizes and shapes has been studied ($0.25 < a/b < 4$, $\beta = 120°$ and 150°). The data (shear along flange and flange area distributions) for $a/b = 2$ and $\beta = 120°$ is presented in Table 1 and Figs. 2 to 5. The convergence of the coefficient A_I which represents the strength of singularity is given in Table 2, for a wide range of panel dimensions and skew angles.

The following symbolic representations of the order and type of solutions are used in Figs. 2 to 5 and these are not explained in the main notations: $A-M=$ Solution PAC $(BD)-M$ and $B-M=$ Solution PBC $(AC)-M$.

V. Discussion of Results

Convergence of Individual Solutions

A careful inspection of the data in Figs. 2 to 5 and Table 1 shows that the stresses due to the individual solutions located at obtuse corner A and acute corner B converge rapidly in large regions (beyond half the flange length) around the respective corners. Each solution by itself requires a large number of terms in the series for accurate data in regions far away from the corner.

Multiple Solutions

In Figs. 2 to 5 the solutions located at obtuse and acute corners are superposed. It is seen that the individual solutions are in good agreement even at first-order approximation in the mid-region, and the region of agreement increase with the order of approximation. Further, near the corners, results from

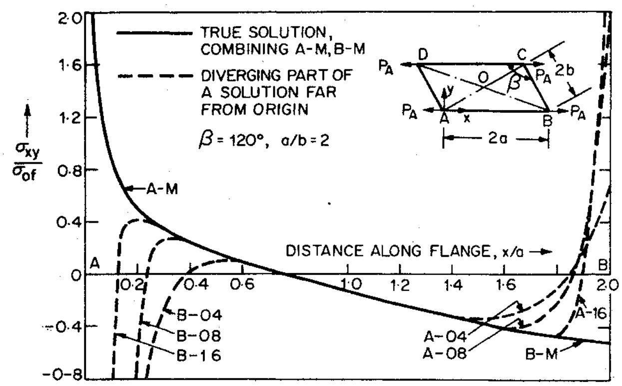

Fig. 2 Shear stress along flange (*AB*) for skew-symmetric loading.

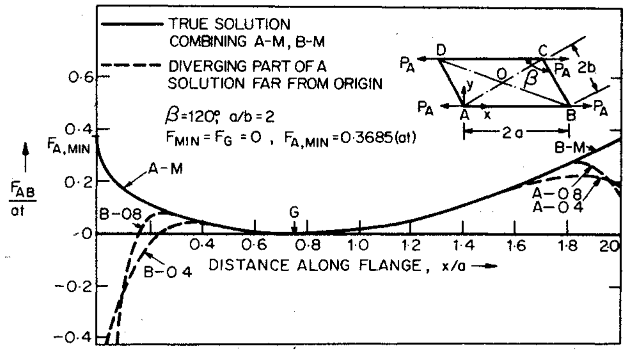

Fig. 3 Flange (*AB*) area distribution for skew-symmetric loading.

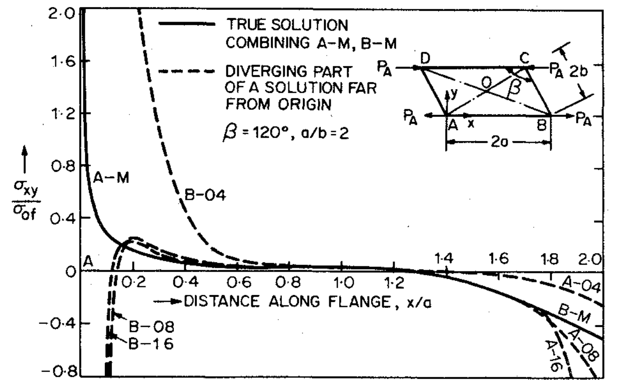

Fig. 4 Shear stress along flange (*AB*) for skew-antisymmetric loading.

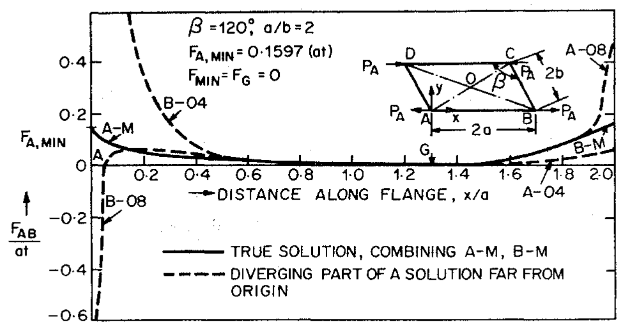

Fig. 5 Flange (*AB*) area distribution for skew-antisymmetric loading.

the respective first-order approximations themselves are accurate. Thus, for practical purposes it is sufficient to draw information from the first-order approximations in the region very close to corner ($x/a<0.5$) and from the second-order approximation in the mid-region ($0.5<x/a<1.5$). This fact is more clearly observed from Table 1. In the figures the true stress and flange area distribution are shown by thick lines. The effort required to draw the true information from first-order approximations is insignificant; in fact, it can be carried out on a desk calculator. It is clear that the combination of solutions is accurate over the entire region, and is more satisfactory than either of the individual solutions.

Limit of Constant Stress in Flanges

The true flange area variations in skew-symmetric and skew-antisymmetric fields are shown in Figs. 3 and 5.

For light weight design, the minimum flange area which occurs at G should, in principle, be made equal to zero ($F_{\min}=F_G=0$). Thus, we obtain the maximum limit for the stress possible in the flange. Specifically, in order to maintain $F_{\min}\geq 0$, in the panel with $a/b=2$, $\beta=120°$, the upper limits of constant stress are ($P_A/0.3685$ at) for skew-symmetric field and ($P_A/0.1597$ at) for skew-antisymmetric field.

Convergence for Different Panel Sizes

Table 2 shows the convergence of the coefficient A_I, which is proportional to the strength of singularity at the obtuse corner. It is seen that the convergence is rapid for a wide range of skew angles ($90°<\beta<150°$), and for side ratios ($0.25<a/b<4.0$).

VI. Concluding Remarks

In this paper a simple method of analysis has been proposed for swept fields. The procedure effectively exploits the skew geometry of the field, and minimizes the computational effort by combining mutiple solutions to cover the field.

In general, for the analysis of fields which cannot be simplified by symmetry concepts, accurate results in the entire fields can often be obtained by combining results of simple analyses, each accurate in a part of the field. It is surprising that this simple procedure of immense practical value does not appear to have been exploited in literature.

References

[1] Williams, M.L., "Stress Singularities Resulting from Various Boundary Conditions in Angular Corners of Plates in Extension," *Journal of Applied Mechanics,* Vol. 19, Dec. 1952, pp. 526-528.

[2] Rao, A.K., "A Study into the Problem of Diffusion of Loads," Ph.D. thesis, 1958, University of London, London, England.

[3] Rao, A.K., "Stress Concentrations and Singularities at Interface Corners," *ZAMM,* Vol. 51, 1971, pp. 395-406.

[4] Mansfield, E.H., "Stress Concentrations at the Cut-outs in a Swept Wing," Rept.& Mem. 2823, July 1971, Aeronautical Research Council (U.K.).

[5] Dattaguru, B. and Rao, A.K., "Stress Concentration and Load Diffusion in Rectangular Panels with Constant Stress Flanges," *Journal of Applied Scientific Reseach,* Vol. 20, May 1969, pp. 326-337.

[6] Hemp, W.S., "On the Application of Oblique Co-ordinates to Problems of Plane Elasticity and Swept-back Wing Structures," Rept., and Mem. 2754, Jan. 1950, Aeronautical Research Council (UK).

[7] Quinlan, P.M., "The λ-method for Skew Plates," *Proceedings of the Fourth U.S. National Congress of Applied Mechanics,* Vol. 2, 1962, pp. 733-751.

[8] Dattaguru, B., "Some Problems of Diffusion and Concentration in Two-dimensional Elasticity," Ph.D. thesis, 1972, Indian Institute of Science, Bangalore, India.

[9] Rao, A.K., Raju, I.S., and Krishna Murty, A.V., "A Powerful Hybrid Method in Finite Element Analysis," *International Journal of Numerical Methods in Engineering,* Vol. 3, No. 3, 1971, pp. 389-403.

[10] Timoshenko, S. and Goodier, J.N., *Theory of Elasticity,* 3rd ed., McGraw-Hill, New York, 1970, pp. 65-68.

[11] Dattaguru, B. amd Ramamurthy, T.S., "Force and Displacement Conditions on Axes of Skew Symmetry," Rept. 317 S, Dept. of Aeronautical Engineering, Indian Institute of Science, Bangalore, India, 1972.